

# Modeling and Performance Comparison of Triple PID and LQR Controllers for Parallel Rotary Double Inverted Pendulum

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**Abstract:** This paper presents the modelling and controller design of nonlinear parallel Rotary Double inverted pendulum system using Proportional-Integral-Derivative (PID) controller and Linear Quadratic Regulator (LQR). LQR, an optimal control technique, and PID control method have been used in this paper to control the nonlinear dynamical system Inverted pendulum, a highly nonlinear unstable system is used as a benchmark for implementing the control methods. In this paper, modelling of the system through Rotational geometry have been carried out, in which the entire inertia tensors are taken into account. The system dynamics for a pendulum with a full inertia tensor using a Lagrangian formulation are presented.

**Keywords:** Nonlinear system, Parallel Rotary Double Inverted Pendulum, Triple PID, LQR, Rotational Geometry.

## I. INTRODUCTION

Inverted pendulum is a well-known benchmark system in control system laboratories which is inherently unstable. In this work full dynamics of the system is derived using classical mechanics and Lagrangian formulation. Dynamic performance are inspected and compared of the two controllers. This paper proves that the LQR controller can promise the inverted pendulum a quick and smoother stabilizing process and with less oscillation and better robustness than the Triple-PID controller. The novelty of this paper is the modelling of the system through rotational geometry, Design and comparison of the two controllers for the Rotary double Inverted pendulum.

## II. MATHEMATICAL MODELING OF PARALLEL ROTARY DOUBLE INVERTED PENDULUM

In this paper the dynamics of the RDIP are derived using the rotational geometry [1] of the system which is explained in section II. Many papers have only considered the rotational inertia of the pendulum for a single principal axis or neglected it altogether [1]. The system dynamics for Rotary Double Inverted Pendulum with a full inertia tensor using a Lagrangian formulation are presented in this paper. Also a linearized model is obtained by neglecting the disturbance torques. Simulation results show the open loop system characteristics

### A. Fundamentals

The physical structure of the RDIP is as shown in Figure 1, where the related physical parameters of the system are listed in the Table 1. The DC motor is used to apply a torque to the rotating arm and the link between rotating arm and Pendulum arm 1 & 2 is not actuated but free to rotate.

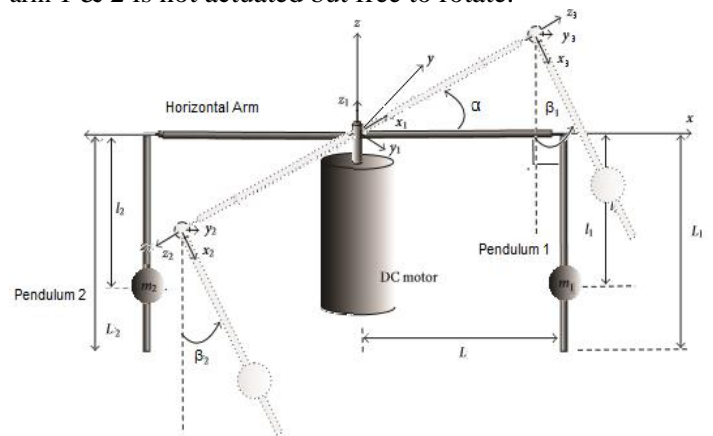


Fig. 1: Schematic of Rotary double inverted pendulum system.

TABLE I. DEFINITION OF PARAMETERS RELATED TO ROTARY DOUBLE INVERTED PENDULUM

Parameter	Definition	Value
m	Total mass of Rotating Arm	0.52Kg
m <sub>1</sub>	Centre of mass of pendulum 1	0.25 Kg
m <sub>2</sub>	Centre of mass of pendulum 2	0.13 Kg
L <sub>1</sub>	Distance from joint to Centre of mass of pendulum1	0.24 m
L <sub>2</sub>	Distance from joint to Centre of mass of pendulum2	0.13 m
m	Mass of the rotating arm	0.52 Kg
L	Length of the rotating arm	0.215 m
b <sub>0</sub>	Viscous coefficients of rotating arm	0.004 N-m-s
b <sub>1</sub>	Viscous coefficients of pendulum1	0.003 N-m-s
b <sub>2</sub>	Viscous coefficients of pendulum2	0.0008 N-m-s
K <sub>m</sub>	Torque constant	0.005 Nm/A
K <sub>b</sub>	Back emf constant	0.001 Volt /rad
R	Resistance of motor circuit	2

A right hand coordinate system has been used to define the rotation of horizontal arm and the two pendulum arms. The coordinate axes of the rotating arm and two pendulum arms are the principal axes such that the inertia tensor are diagonal in the form as in (1).

$$\begin{aligned}
 J_1 &= \begin{pmatrix} J_{x_1x} & 0 & 0 \\ 0 & J_{y_1y} & 0 \\ 0 & 0 & J_{z_1z} \end{pmatrix} \\
 J_2 &= \begin{pmatrix} J_{x_2x} & 0 & 0 \\ 0 & J_{y_2y} & 0 \\ 0 & 0 & J_{z_2z} \end{pmatrix} \\
 J_3 &= \begin{pmatrix} J_{x_3x} & 0 & 0 \\ 0 & J_{y_3y} & 0 \\ 0 & 0 & J_{z_3z} \end{pmatrix}
 \end{aligned} \tag{1}$$

$J_1, J_2, J_3$  indicate the Inertia tensors of Rotating arm, Pendulum1, Pendulum2 respectively.  $J_{x_1x}$ , indicate rotation of the arm along x axis when a torque is applied along  $x_1$  axis. Similarly we can define  $J_{y_1y}$  and  $J_{z_1z}$ . The angular rotation of rotating arm, is measured in the horizontal plane where a counterclockwise direction (when viewed from above) is positive [1]. The angular rotation of pendulum 1,  $\theta_1$ , is measured in the vertical plane where a counterclockwise direction (when viewed from the front) is positive, same as in pendulum 2 is  $\theta_2$ , when pendulums are hanging down in the stable equilibrium position  $\theta_1 = \theta_2 = 0$ .  $\tau_1, \tau_2$  are the disturbance torque experienced by pendulum1 and pendulum2 respectively.

Energies of the arms are defined as:

For rotating arm,

Potential energy is given by

$$P_0 = 0$$

Kinetic energy is,

$$\begin{aligned}
 k_0 &= \frac{1}{2} (V_1^T m V_1 + \dot{S}_1^T J_1 \dot{S}_1) \\
 &= \frac{1}{2} r^2 (mL^2 + J_{z_1z})
 \end{aligned} \tag{2}$$

Potential energy of pendulum1 is given by

$$P_1 = m_1 g l_1 \cos(S_1) \tag{3}$$

And Kinetic energy is,

$$\begin{aligned}
 k_1 &= \frac{1}{2} (V_2^T m_1 V_2 + \dot{S}_2^T J_2 \dot{S}_2) \\
 &= \frac{1}{2} m_1 (r^2 L^2 \sin^2(S_1) + (r L \cos(S_1) + l_1 S_1)^2 \\
 &\quad + r^2 l_1^2 \sin^2(S_1)) + \frac{1}{2} (r^2 J_{x_2x} \cos^2(S_1) \\
 &\quad + r^2 \sin^2(S_1) J_{y_2y} + S_1^2 J_{z_2z})
 \end{aligned} \tag{4}$$

Potential energy of pendulum 2 is given by

$$P_2 = m_2 g l_2 \cos(S_2) \tag{5}$$

Kinetic energy is given by

$$\begin{aligned}
 K_2 &= \frac{1}{2} (V_3^T m_2 V_3 + \dot{S}_3^T J_3 \dot{S}_3) \\
 &= \frac{1}{2} m_2 (r^2 L^2 \sin^2(S_2) + (r L \cos(S_2) + l_2 S_2)^2 + r^2 l_2^2 \sin^2(S_2)) \\
 &\quad + \frac{1}{2} (r^2 J_{x_3x} \cos^2(S_2) + r^2 \sin^2(S_2) J_{y_3y} + S_2^2 J_{z_3z})
 \end{aligned} \tag{6}$$

Total potential energy,

$$P_{total} = P_1 + P_2 + P_0$$

Total kinetic energy,

$$K_{total} = K_0 + K_1 + K_2$$

We can write Lagrange equation  $L = KE - PE$  as follows

$$\begin{aligned}
 L &= \frac{1}{2} r^2 (mL^2 + J_{z_1z}) + \frac{1}{2} m_1 (r^2 L^2 \sin^2(S_1) + (r L \cos(S_1) + l_1 S_1)^2 \\
 &\quad + r^2 l_1^2 \sin^2(S_1)) + \frac{1}{2} (r^2 J_{x_2x} \cos^2(S_1) + r^2 \sin^2(S_1) J_{y_2y} + S_1^2 J_{z_2z}) \\
 &\quad + \frac{1}{2} m_2 (r^2 L^2 \sin^2(S_2) + (r L \cos(S_2) + l_2 S_2)^2 + r^2 l_2^2 \sin^2(S_2)) + \\
 &\quad \frac{1}{2} (r^2 J_{x_3x} \cos^2(S_2) + r^2 \sin^2(S_2) J_{y_3y} + S_2^2 J_{z_3z}) - m_2 g l_2 \cos(S_2) \\
 &\quad - m_1 g l_1 \cos(S_1)
 \end{aligned} \tag{7}$$

The velocities of the pendulum arms are derived based on the rotation matrices and frame of references defined for each arm.

Since the arms are long and slender moment of inertia is considered to be negligible, more over the arms have rotational symmetry such that the moment of inertia in the principal axes are equal thus inertia tensor can be approximated as follows

$$\begin{aligned}
 J_1 &= \begin{pmatrix} J_{x_1x} & 0 & 0 \\ 0 & J_{y_1y} & 0 \\ 0 & 0 & J_{z_1z} \end{pmatrix} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_1 \end{pmatrix} \\
 J_2 &= \begin{pmatrix} J_{x_2x} & 0 & 0 \\ 0 & J_{y_2y} & 0 \\ 0 & 0 & J_{z_2z} \end{pmatrix} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{pmatrix} \\
 J_3 &= \begin{pmatrix} J_{x_3x} & 0 & 0 \\ 0 & J_{y_3y} & 0 \\ 0 & 0 & J_{z_3z} \end{pmatrix} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_3 \end{pmatrix}
 \end{aligned} \tag{8}$$

Above dynamic equation can be written in a little easier way by making the following substitution:

Total moment of inertia of rotating arm about the pivot point,

$$\hat{J}_1 = J_1 + mL^2 \tag{9}$$

Moment of inertia of pendulum1 about its pivot point,

$$\hat{J}_2 = J_2 + m_1 l_1^2 \tag{10}$$

Moment of inertia of pendulum2,

$$\hat{J}_3 = J_3 + m_2 l_2^2 \tag{11}$$

Total moment of inertia experience by the motor when pendulum 1 and 2 are in hanging position,

$$\begin{aligned} \hat{J}_0 &= \hat{J}_1 + m_1 L^2 + m_2 L^2 \\ &= J_1 + mL^2 + m_1 L^2 + m_2 L^2 \end{aligned} \tag{12}$$

Initially the control input is the torque  $\tau_1$  applied to the pivot of the rotating arm. DC motor is used to drive the rotating arm, so the voltage is taken as the control input. As far as we are neglecting the effect of inductor, the torque and the voltage can be related by the equation,

$$\ddagger = \frac{K_m V}{R} - \frac{K_m K_b r}{R} \tag{13}$$

Where  $K_m, K_b, R$  are the motor parameters mentioned in the Table 1. A compact form can be obtained by using these simplifications, thus obtaining a coupled electro mechanical equation as follows,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{bmatrix} r \\ S_1 \\ S_2 \end{bmatrix} + \begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} = \begin{pmatrix} \frac{K_m}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} V \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{14}$$

Where,

$$\begin{aligned} A_{11} &= \hat{J}_0 + m_1 l_1^2 \sin^2(S_1) + m_2 l_2^2 \sin^2(S_2) \\ &\quad + J_2 \sin^2(S_1) + J_3 \sin^2(S_2) \\ A_{12} &= m_1 l_1 L \cos(S_1) \\ A_{13} &= m_2 l_2 L \cos(S_2) \\ A_{21} &= m_1 l_1 L \cos(S_1) \\ A_{22} &= \hat{J}_2 \\ A_{23} &= 0 \\ A_{31} &= m_2 l_2 L \cos(S_2) \\ A_{32} &= 0 \\ A_{33} &= \hat{J}_3 \\ A'_1 &= -m_1 l_1 L \sin(S_1) S_1^2 + m_1 l_1^2 r S_1 \sin(2S_1) \\ &\quad - m_2 l_2 L \sin(S_2) S_2^2 + m_2 l_2^2 r S_2 \sin(2S_2) \\ &\quad + J_2 r S_1 \sin(2S_1) + J_3 r S_2 \sin(2S_2) + (b_0 + \frac{K_m K_b}{R}) r \\ A'_2 &= -\frac{1}{2} r^2 \sin(2S_1) \hat{J}_2 - m_1 l_1 g \sin(S_1) + b_1 S_1 \\ A'_3 &= -\frac{1}{2} r^2 \sin(2S_2) \hat{J}_3 - m_2 l_2 g \sin(S_2) + b_2 S_2 \end{aligned} \tag{15}$$

**B. Linearized state equation**

Linearized equation of the dynamic system for the upright position is derived below.

Finding the linearized model using the equilibrium point where  $\beta_1 = \beta_2 = 180^\circ$ . After expanding the Taylor series at  $x = \pi$

$$\sin(x) \approx \pi - x, \sin^2(x) \approx 0, \cos(x) \approx -1$$

We can make approximation of the nonlinear equation in (14) around the equilibrium point  $\beta_1 = \beta_2 = 180^\circ$  thus equation becomes,

$$\begin{aligned} \hat{J}_0(\ddot{r}) - m_1 l_1 L S_1 - m_2 l_2 L S_2 + (b_0 + \frac{K_m K_b}{R}) r &= \frac{K_m V}{R} \\ -m_1 l_1 L(\ddot{r}) + \hat{J}_2 \ddot{S}_1 + m_1 l_1 g(S_1 - f) + b_1 S_1 &= \ddagger_2 \\ -m_2 l_2 L(\ddot{r}) + \hat{J}_3 \ddot{S}_2 + m_2 l_2 g(S_2 - f) + b_2 S_2 &= \ddagger_3 \end{aligned} \tag{16}$$

Here  $x$  can be taken as  $x = [r \ S_1 - f \ S_2 - f \ r \ S_1 \ S_2]^T$  be the state variable and control input be the voltage,  $v$ . Then (16) can arrange in the form of

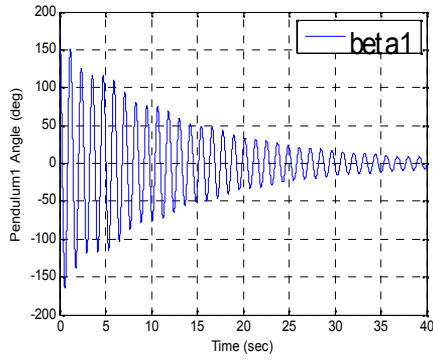
$$\begin{aligned} P x &= Q x + R u \\ x &= P^{-1} Q x + P^{-1} R u =: A x + B u \end{aligned} \tag{17}$$

$$\begin{aligned} \text{Where,} \\ P &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{J}_0 & -m_1 l_1 L & -m_2 l_2 L \\ 0 & 0 & 0 & -m_1 l_1 L & \hat{J}_2 & 0 \\ 0 & 0 & 0 & -m_2 l_2 L & 0 & \hat{J}_3 \end{pmatrix} \\ Q &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -(b_0 + \frac{K_m K_b}{R}) & 0 & 0 & 0 \\ 0 & -m_1 g l_1 & 0 & 0 & 0 & -b_1 & 0 \\ 0 & 0 & -m_2 g l_2 & 0 & 0 & 0 & -b_2 \end{pmatrix} \\ R &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{K_m}{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{18}$$

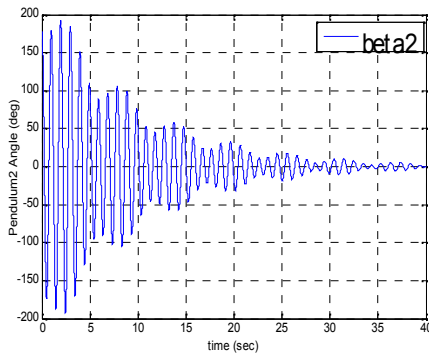
Disturbance torque may be neglected in the analysis, so the R matrix became

$$R = [0 \ 0 \ 0 \ \frac{K_m}{R} \ 0 \ 0]^T \tag{19}$$

Open loop responses of the system under various initial conditions are given in the fig 2



(a)



(b)

Fig 2. (a) Open loop response of  $\beta_1(0)=175^\circ$  (b) Open loop response of  $\beta_2(0)=178^\circ$

**III. CONTROLLER DESIGN**

In this paper Triple PID and LQR control strategies has been used to stabilize parallel rotary inverted pendulum. PID control make use the effects of the proportional ( $K_p$ ), derivative ( $K_d$ ) and integral ( $K_i$ ) control. Sometimes PID controllers does not provide sufficient performance for the control of the nonlinear and undefined systems. An optimal control system is designed to balance the same designed successfully using LQR. This method can be adopted either to linear system or rather linearized system.

*A. Designing and Tuning of Triple PID controller*

In literature many method have been proposed to control the inverted pendulum, such as traditional PID control [2], fuzzy control [3], genetic algorithm optimizing control [4], and linear quadratic regulator (LQR) control [5]. Even though many control algorithms are proposed for the control of inverted pendulum, PID control is the most widely used control method in the realization of the control system. However, the rotary double inverted pendulum system is a one input and three output system which challenges to the one input and one output control characteristic of the single PID controller. While we consider inverted pendulum, having two output, only one can be controlled and mostly cart position control will be neglected. In this paper, we made an attempt to solve this multi-output problem by implementing a triple PID controller.

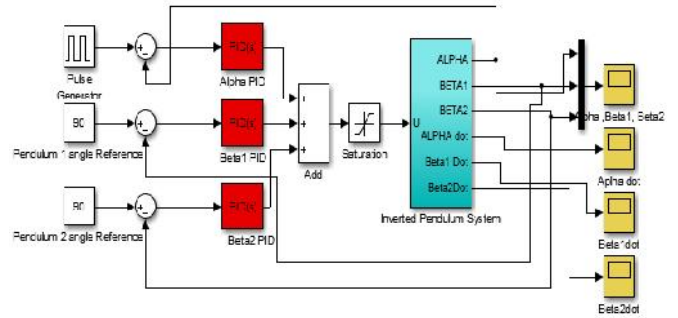


Fig 3. Block Diagram of Triple PID controller for parallel Rotary Double Inverted Pendulum

In this proposed work the reference angle is taken as 90 deg.[6] Pulse signal is used as the position reference signal, with an amplitude of 0.5m, pulse period 20s and width 50% of period, by which the ability of the controller to track the position is being checked. After tuning the control parameters, we obtain a group of parameters for double-PID controller, with which the controller is capable of controlling the inverted pendulum system and providing a very robust performance.

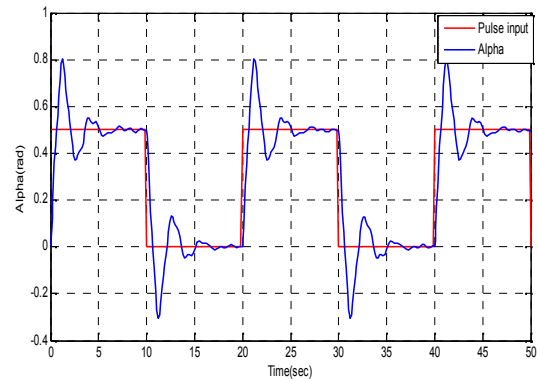


Fig 4: Horizontal arm angle when tracking pulse signal.

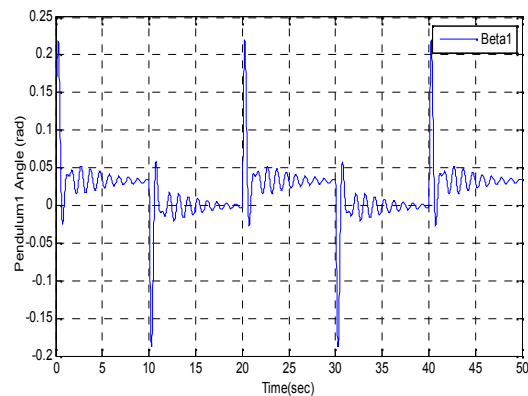


Fig 5. Pendulum1 angle keep balancing while Horizontal Arm tracking reference signal  
After tuning, the control parameter are obtained as

$$[K_{pr} = 80.23, K_{ir} = 50.021, K_{Dr} = 40]$$

$$[K_{ps_1} = 356.28, K_{is_1} = 120.36, K_{Ds_1} = 150]$$

$$[K_{ps_2} = 232.16, K_{is_2} = 102, K_{Ds_2} = 132.56]$$

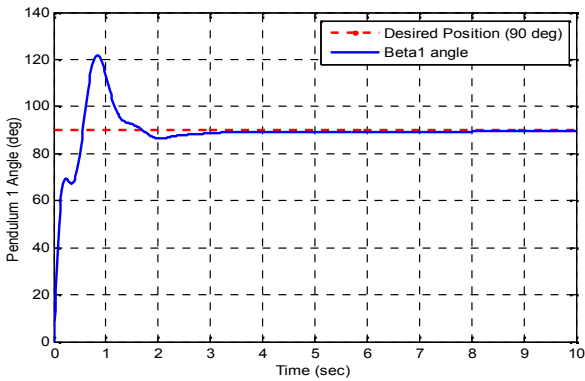


Fig 6. PID control of pendulum 1 Angle

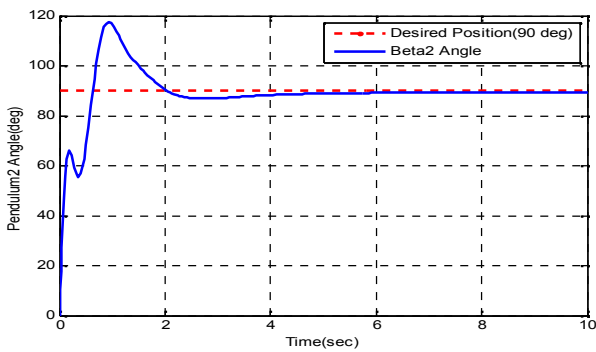


Fig. 7 PID control of pendulum 2 Angle

**B. Designing and tuning of LQR controller**

The plant is linear time invariant and the state space equation is

$$\dot{X} = AX + BU \tag{20}$$

Which minimize the performance index cost

$$J = \int_0^{\infty} (X^T QX + U^T RU) dt \tag{21}$$

Where Q is a semi definite matrix, R is a positive definite matrix. State variable and input variables are penalized by Q and R matrices respectively for the smallest performance index function. To minimize the cost function the feedback control law is obtained as

$$u(t) = -kx(t) = R^{-1}B^T Px(t) \tag{22}$$

Where P is the only positive definite symmetric solution which meets the Riccati equation

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \tag{23}$$

The optimality of the LQR control algorithm totally depends on choosing of Q and R matrix. However, there is no resolving method to choose these two matrices. The usual method to choose Q and R is by means of Simulation and trial.[8] In this section, the LQR controller is tuned by changing the nonzero elements of the Q matrix elements,

which is closely influenced by the three concerned states: Horizontal arm angle and pendulum angle1 and pendulum angle 2. After trial and error we choose the weighting matrix as and corresponding feedback gain matrix as

$$Q = \text{diag} [400, 800, 100, 0, 100, 200], \text{ and } R=1$$

$$K = [2 -500 800 30.97 203 36.5]$$

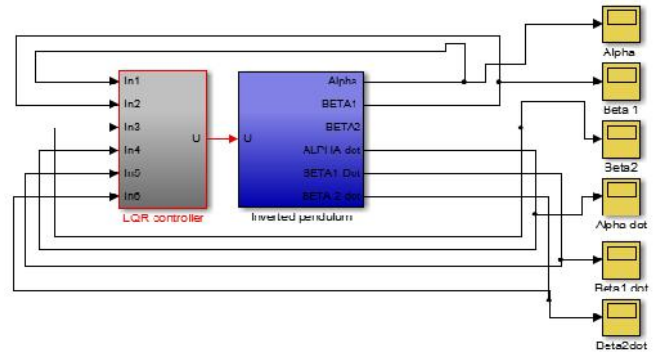
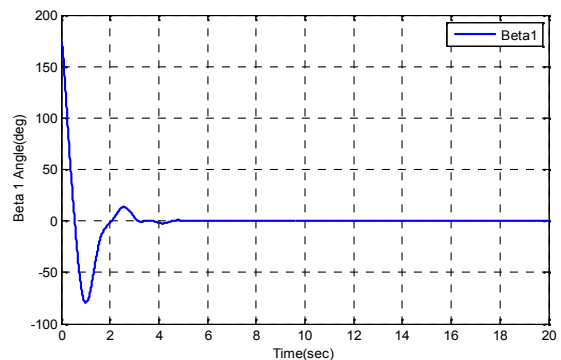


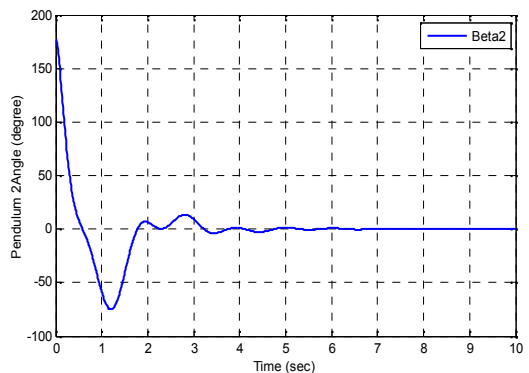
Fig 8. Block diagram of the designed LQR controller

**C. Assessment of Dynamic response using designed Controllers**

Dynamic performance indices are chosen to reflect practical control effect. Performance indices include *rise time*, *transition time*, *steady state error*, *peak overshoot*.



(a)



(b)

Fig 9. Dynamic response of the system using LQR for (a)  $\theta_1(0)=175^\circ$  (b)  $\theta_2(0)=178^\circ$

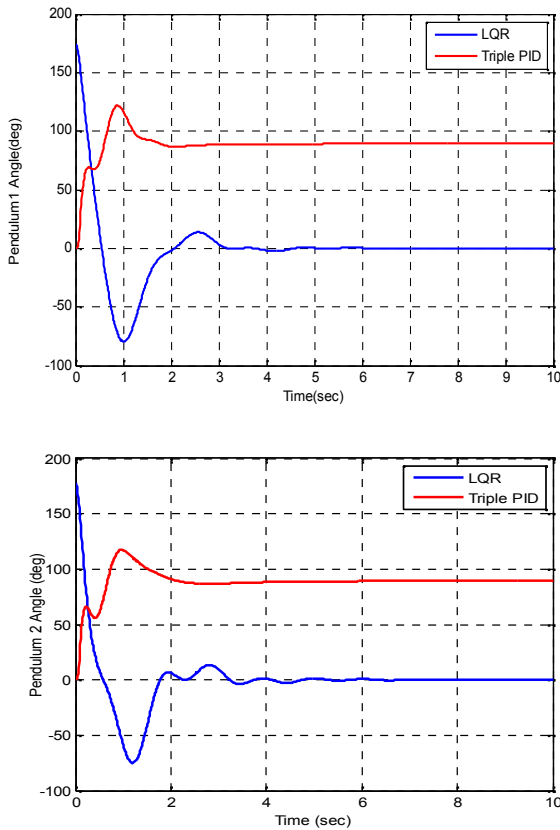


Fig 10. Dynamic response comparison of designed controllers

#### IV. CONCLUSION

A Triple PID and LQR, an optimal control technique have been implemented in this paper to control parallel Rotary Double inverted pendulum. The dynamic response of the system for each controller have been compared. Simulations for the control schemes have been carried out using MATLAB-SIMULINK models. The simulation results justify the comparative advantages of optimal control using LQR method than Triple PID control. Dynamic performance proves that the proposed LQR controller can guarantee the Rotary Double Inverted pendulum a faster and smoother stabilizing process and with better robustness than the Triple-PID controller.

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